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On scheme dependence of gravitational dressing of renormalization group functions

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Abstract

It is shown that for 2D field theories only the first order coefficient of the gravitationally dressed RG β-function is scheme independent. This is valid even for matter theories with one dimensionless coupling, where the first two coefficients of the original β-function are scheme independent.

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For the gravitational dressing of renormalization group β-functions of two-dimensional field theories there has been established in *lowest* order of perturbation theory the remarkable universal formula [1, 2, 3, 4, 5]

$$\bar{\beta}(g) \;=\; b(c)\cdot\beta(g)\;. \tag{1}$$

The factor $b(c)$ depends only on the central charge c of the conformally invariant theory around which one is perturbing

$$b(c) \;=\; \frac{2}{\alpha Q}\;, \qquad Q^2 \;=\; \frac{25-c}{3}\;, \qquad \alpha \;=\; \frac{Q-\sqrt{Q^2-8}}{2}\;. \tag{2}$$

A simple argument [6] based on the c-theorem [7] shows that (1) cannot be valid to all orders. Therefore, it is an interesting question to ask for the higher orders. But then in general one has to face the problem of scheme dependence. A natural candidate to find a scheme independent universal formula [2] seemed to be theories with only one coupling constant. In the absence of a linear term the first two coefficients of their β are scheme independent. A corresponding formula was derived in [6] having in mind the prejudice that scheme independence holds also for the corresponding dressed $\bar{\beta}$ coefficients.

Meanwhile, there has been found [8] a second order formula based on a completely different approach [9], which differs from our result by a factor 2 in a relative weight. One explanation of such a difference would be scheme dependence of the dressing procedure [10]. Therefore, the first aim of this note is to analyze from the very beginning the question of scheme dependence within our own approach. The analysis of [6] also involved manipulations with a divergent bare coupling. As a byproduct of our improved treatment we will be able to avoid these formal steps.

Throughout this letter we use the notations and conventions of [6]. In our discussion three β-functions appear. $\beta(g)$ denotes the β-function of the original matter theory without gravity, $\tilde{\beta}(g)$ is the β-function of the combined matter-gravity system refering to scaling in the (unphysical) coordinate space. The condition $\tilde{\beta}(g)=0$ fixes the gravitational dressing. Finally the gravitationally dressed function $\bar{\beta}(g)$ is connected to the response of the system to a change in the cosmological constant which sets the physical scale after coupling to quantum gravity.

Our starting point is the gravitationally dressed action

$$\begin{aligned}\tilde{S} \;&=\; S_c \;+\; S_L \;+\; \sum_i g_i \int \tilde{V}_i\,\sqrt{\hat{g}}\;d^2z\;,\\ S_L[\phi|\hat{g}] \;&=\; \frac{1}{8\pi}\int d^2z\sqrt{\hat{g}}\Big(\hat{g}^{mn}\partial_m\phi\partial_n\phi + Q\hat{R}\phi(z) + m^2e^{\alpha\phi}\Big)\;.\end{aligned} \tag{3}$$

S_c denotes the conformal matter action. The 2D metric is treated in conformal gauge $g_{ab}=e^{\alpha\phi}\,\hat{g}_{ab}$. The Liouville mass parameter m^2 plays the role of the cosmological constant.

[2] By universal we understand a formula which relates the coefficients of the power series of $\bar{\beta}$ to that of β and which requires no information beyond β and c.

For the relation between the dressed and undressed perturbation the ansatz

$$\tilde{V}(z) \;=\; e^{\delta\phi(z)} \cdot V(z) \tag{4}$$

is made. The dimension of $\tilde{V}$ is $2-y$ with

$$y \;=\; \delta(\delta - Q) \;. \tag{5}$$

We extract the renormalization Z-factor needed to calculate $\tilde{\beta}$ from the two-point function of $\tilde{V}(z)$. $y>0$ acts as a regularization parameter, effectively. From

$$\langle\tilde{V}(z_1)\tilde{V}(z_2)\rangle \;=\; \langle\tilde{V}(z_1)\tilde{V}(z_2)\rangle_0 \;-\; g\int d^2z\langle\tilde{V}(z_1)\tilde{V}(z_2)\tilde{V}(z)\rangle_0 \;+\; O(g^2) \tag{6}$$

together with

$$\begin{aligned}
\langle\tilde{V}(z_1)\tilde{V}(z_2)\rangle_0 \;&=\; \left(\frac{m^2}{\bar{\mu}^2}\right)^{\frac{Q-2\delta}{\alpha}} B_2(\delta)\;|z_1-z_2|^{2y-4}\;, \\
\langle\tilde{V}(z_1)\tilde{V}(z_2)\tilde{V}(z_3)\rangle_0 \;&=\; f\cdot\left(\frac{m^2}{\bar{\mu}^2}\right)^{\frac{Q-3\delta}{\alpha}} B_3(\delta,\delta,\delta)\;(|z_1-z_2||z_1-z_3||z_2-z_3|)^{y-2}
\end{aligned} \tag{7}$$

we get

$$\begin{aligned}
\langle\tilde{V}(z_1)\tilde{V}(z_2)\rangle \;&=\; \left(\frac{m^2}{\bar{\mu}^2}\right)^{\frac{Q-2\delta}{\alpha}} B_2(\delta)\;|z_1-z_2|^{2y-4} \\
&\quad -\; \frac{\pi g f B_3(\delta,\delta,\delta)}{|z_1-z_2|^{4-3y}}\left(\frac{m^2}{\bar{\mu}^2}\right)^{\frac{Q-3\delta}{\alpha}} \frac{\Gamma(1-y)(\Gamma(\frac{y}{2}))^2}{\Gamma(y)(\Gamma(1-\frac{y}{2}))^2} \;+\; O(g^2)\;.
\end{aligned} \tag{8}$$

The constant f parametrizes the 3-point function of the undressed V. [3]

$$A_N \;=\; \left(\frac{m^2}{\bar{\mu}^2}\right)^{\frac{Q-N\delta}{\alpha}} B_N\;, \qquad N=2,\;3 \tag{9}$$

is the z-independent part of the 2- and 3-point function for exponentials of the Liouville field ϕ [11, 12]. In contrast to [6] we keep track of the m dependence introduced by the Liouville sector more carefully, B_N contains no further m dependence. After factorizing in (8) an overall factor A_2 we see that in the remaining part m only appears in the combination

$$g\;\left(\frac{m^2}{\bar{\mu}^2}\right)^{-\frac{\delta}{\alpha}}\;.$$

Although the N-point functions of Liouville exponentials for $N>3$ have not been constructed explicitly, the last statement on m dependence is valid for all N [13].

[3] $\bar{\mu}$ is a scale needed to form a dimensionless ratio. For a complete analysis of all dimensionful parameters involved see [13].

To take care of the 2D coordinate space dimension of the bare coupling g we introduce a RG-scale μ which has to be distinguished from $\bar{\mu}$ and define the dimensionless Z-factor for the coupling constant renormalization by

$$g\left(\frac{m^2}{\bar{\mu}^2}\right)^{-\frac{\delta}{\alpha}} = \mu^y \; Z_g(y,g_r)\; g_r \; . \tag{10}$$

This yields

$$\tilde{\beta} = \mu\frac{\partial}{\partial\mu}g_r\Big|_{g,m,\bar{\mu},y \text{ fix}} = \frac{-y\; g_r}{1 + g_r\frac{\partial \log Z_g}{\partial g_r}} \; . \tag{11}$$

Besides Z_g the renormalization of (8) involves the Z-factor of the composite operator $\tilde{V}$

$$\tilde{V} = \mu^{-y}\left(\frac{m^2}{\bar{\mu}^2}\right)^{-\frac{\delta}{\alpha}} Z_{\tilde{V}}(y,g_r)\tilde{V}_r \; . \tag{12}$$

Due to $\int d^2z\sqrt{\hat{g}}\tilde{V}_r = \frac{\partial\tilde{S}}{\partial g_r}$ it is related to Z_g by

$$Z_{\tilde{V}}^{-1} = Z_g + g_r\frac{\partial Z_g}{\partial g_r} \; . \tag{13}$$

Now the renormalized two-point function $\langle\tilde{V}(z_1)\tilde{V}(z_2)\rangle_r = \mu^{2y}\left(\frac{m^2}{\bar{\mu}^2}\right)^{\frac{2\delta}{\alpha}} Z_{\tilde{V}}^{-2}\langle\tilde{V}(z_1)\tilde{V}(z_2)\rangle$ expressed in terms of g_r has to be finite. With (8), (13), (10) this implies [4]

$$Z_g = 1 + \left(\frac{\pi f}{y}+\gamma\right) g_r + O(g_r^2) \; . \tag{14}$$

γ is an arbitrary number. In contrast to [6] we do not restrict ourselves to minimal subtraction with respect to the 'y-regularization'. From (11) we get the corresponding β-function $\tilde{\beta}$

$$\tilde{\beta}(y,g_r) = -yg_r + (\pi f+\gamma y)g_r^2 + \tilde{\beta}_3(y)g_r^3 + \ldots \; . \tag{15}$$

Of course the deviation from minimal subtraction parametrized by γ has no influence on the first two nonvanishing coefficients of the original matter β-function $\beta(g_r)$ which is obtained for $y\to 0$

$$\beta_2 = \tilde{\beta}_2(0) = \pi f \; , \qquad \beta_3 = \tilde{\beta}_3(0) \; . \tag{16}$$

However, the freedom to choose γ influences the dependence of y on g_r imposed by the condition $\tilde{\beta}(y,g_r) = 0$. This condition ensures independence of the unphysical scale in coordinate space and fixes the gravitational dressing of the perturbation V within our ansatz (4)

$$\tilde{\beta}(y,g_r) = 0 \qquad \Leftrightarrow \qquad y = y(g_r) = \beta_2 g_r + (\beta_3+\beta_2\gamma)g_r^2 + O(g_r^3) \; . \tag{17}$$

[4] As discussed in [6] we use $B_3/B_2 = 1 + O(y)$.

Via (5) this leads to

$$\delta \;=\; \delta(g_r) \;=\; -\;\frac{\beta_2}{Q} g_r \;+\; \frac{1}{Q}\left(\frac{\beta_2^2}{Q^2} - \beta_3 - \beta_2\gamma\right) g_r^2 \;+\; O(g_r^3)\;. \tag{18}$$

Before we proceed to the construction of the gravitational dressed $\bar{\beta}$ we have to have a closer look on the condition $\tilde{\beta} = 0$. Although for general $y > 0$ the bare coupling is finite, eq. (11) tells us that $\tilde{\beta} = 0$ at finite $y \neq 0$ can be achieved for the price of infinite Z_g i.e. infinite g only. This effect is in complete analogy to the situation at the nontrivial fixed point in Φ^4 theory in $4 - \epsilon$ dimensions [14]. There it has been understood and used to derive scaling relations not at but near the critical temperature for distances x very large (very small) compared to the lattice constant (correlation length).

Let us look at (10) in the form

$$g Z_g^{-1}(y, g_r)\mu^{-y} \;=\; \left(\frac{m^2}{\bar{\mu}^2}\right)^{\frac{\delta}{\alpha}} g_r\;. \tag{19}$$

Although both g and Z_g diverge for $y \to y(g_r)$, $\delta \to \delta(g_r)$ both sides of (19) remain finite in this limiting process. Since in addition the m-dependence in a factorized way is completely decoupled from the divergence just discussed, it is naturally to define

$$\bar{g} \;=\; g Z_g^{-1}(y, g_r)\mu^{-y} \tag{20}$$

as the bare coupling with respect to a change of m. Then

$$\bar{\beta}(g_r) \;=\; m\frac{\partial}{\partial m} g_r\Big|_{\bar{g},\ \bar{\mu}\ \text{fix}} \tag{21}$$

delivers us

$$\begin{aligned}\bar{\beta}(g_r, \frac{m}{\bar{\mu}}) \;&=\; \frac{2\beta_2}{\alpha Q}\, g_r^2 \;+\; \frac{2}{\alpha Q}\left(\beta_3 + \gamma\beta_2 - \frac{\beta_2^2}{Q^2}\right)\, g_r^3 \\ &+\; \frac{4\beta_2^2}{\alpha^2 Q^2}\, g_r^3\, \log\frac{m}{\bar{\mu}} \;+\; O(g_r^4)\;. \end{aligned} \tag{22}$$

This coincides with [6] for $\gamma = 0$.

In conclusion we want to stress that the origin of scheme dependence of second order coefficients for the dressed RG-function $\bar{\beta}$ can be traced back to the presence of a linear term in $\tilde{\beta}$. It does not spoil scheme independence of the first two coefficients of the original β since this function is related to the $y \to 0$ limit where the linear term drops out. However, the gravitational dressing fixed by $\tilde{\beta} = 0$ requires $y \neq 0$. An alternative point of view would be to interpret y or δ not as a regularization parameter but as a second coupling. In theories with more than one coupling the second order contribution is scheme dependent.

As we have seen there is no universal formula relating $\bar{\beta}$ to β beyond lowest order. Nevertheless, if one lowers the demands on universal validity, within a given renormalization scheme one can look for expressions for the coefficients of $\bar{\beta}$ in terms of that of β which as additional information involve c only. Then our formula (22) for $\gamma = 0$ yields the answer for minimal subtracted 'y-regularization'. It would be interesting to derive a similar result in a more standard scheme as e.g. minimal subtracted dimensional regularization.

The divergence of the original bare coupling (related to the RG with respect to the scaling in the unphysical coordinate space) in the presence of a regularization, which is forced by $\tilde{\beta}(y, g_r) = 0$, requires the definition of a new bare coupling $\bar{g}$ as in (20). The similarity with an effect in the theory of critical phenomena is remarkable.

A last comment concerns the $\log \frac{m}{\bar{\mu}}$ term in (22). In contrast to the β-function in the MOM-scheme in standard field theories (see e.g. [15]) our m-dependence exhibits no threshold behaviour. We expect this to be related to the peculiar role of m in Liouville theory. A scaling of m can be compensated by a shift of the constant mode of the Liouville field.

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